INPUT SPARSITY TIME LOW-RANK APPROXIMATION VIA RIDGE LEVERAGE SCORE SAMPLING

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Supported by new results in random matrix theory and understanding of how to use these results algorithmically.

Closely tied to work on graph sparsification, fast laplacian solvers, streaming algorithms, compressed sensing, etc.
[Clarkson Woodruff STOC ’13]: Sparse Random Projections
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\[\text{Solution for } A \quad \text{approximate solution for } \tilde{A} \]

- For problems like linear system solving, low-rank approximation, etc.
- \( O(\|n\| \|d\|) \) to compute \( A \) plus lower order terms = input sparsity time \( 2 \).
[Clarkson Woodruff STOC ’13]: Sparse Random Projections

- Solution for $\tilde{\mathbf{A}} \Rightarrow$ approximate solution for $\mathbf{A}$ for problems like linear system solving, low-rank approximation, etc.
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- $O(\text{nnz}(A))$ to compute $A\Pi$ plus lower order terms = input sparsity time
Set $Q \leftarrow$ top $k$ left singular vectors of $\tilde{A}$. 

\[
\|AQQ^TA\|_F (1 + \epsilon) \min_{B} \text{rank}(B) = k \\|AB\|_F
\]

$O(k/\epsilon^2)$
Set $Q \leftarrow \text{top $k$ left singular vectors of } \tilde{A}.$

$$\|A - QQ^T A\|_F^2 \leq (1 + \epsilon) \min_{B \mid \text{rank}(B) = k} \|A - B\|_F^2$$
Runtime:
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- \( O(nk^2/\epsilon^4) \) time to compute \( \tilde{A} \)'s top singular vectors.
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Total: $O(\text{nnz}(A)) + n \cdot \text{poly}(k, 1/\epsilon)$

[lower order]
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- Many improvements. See [Avron Clarkson Woodruff ’16] for best low order terms.
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**Total:** \( O(\text{nnz}(A)) + n \cdot \text{poly}(k, 1/\epsilon) \)

- Many improvements. See [Avron Clarkson Woodruff '16] for best low order terms.
- Compare with \( \tilde{O}(\text{nnz}(A) \cdot k/\sqrt{\epsilon}) \) for iterative methods.
Main Result: Input sparsity time low-rank approximation without sparse random projections.
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- Column subset selection in single-pass streams.
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- Column subset selection in single-pass streams.
- Linear time algorithms for Nyström kernel approximation [Musco Musco ’16].
- Sublinear time, relative error algorithms for low-rank approximation of PSD matrices [Musco Woodruff ’16]
DIMENSIONALITY REDUCTION VIA IMPORTANCE SAMPLING

Extremely simple and efficient... once $S$ is known.
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$A \rightarrow \tilde{A}$

$O(k/\varepsilon^2)$
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1. Brief discussion of techniques
2. Why care about sampling?
Leverage scores are the natural sampling probabilities for relative error matrix approximation.
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**Intuition:** Measure **uniqueness** of column. \( \tau(a_i) = \min ||y||_2^2 \) such that \( Ay = a_i. \)
LEVERAGE SCORE SAMPLING

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Sampling \( \tilde{O}(\text{rank}(A)/\epsilon^2) \) columns by leverage scores gives spectral approximation:

\[
(1 - \epsilon)AA^T \preceq \tilde{A}\tilde{A}^T \preceq (1 + \epsilon)AA^T.
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LEVERAGE SCORE COMPUTATION

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- Leads to \(O(\text{nnz}(A))\) time recursive sampling algorithm for leverage score approximation [Cohen, Lee, Musco, Musco, Peng, Sidford ’15].
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- Leads to $O(\text{nnz}(A))$ time recursive sampling algorithm for leverage score approximation [Cohen, Lee, Musco, Musco, Peng, Sidford ’15].
- Input sparsity time regression without sparse projections.
“Subspace Scores” [Drineas, Mahoney, Muthukrishnan ’08], [Sarló ’06]:

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where \( A_k = \arg\min_B \|A - B\|_F \) 

for rank-B = k.
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where \( A_k = \arg \min_B \|A - B\|_F \) s.t. \( \text{rank}(B) = k \).

- Gives additional error depending on \( \|A - A_k\|_F \Rightarrow \text{good enough for near optimal low-rank approximation.} \)
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- But this is what we want to compute in the first place! Hence all \( \text{nnz}(A) \) time sampling algorithms rely critically on sparse random projections.

Further, subspace scores are unstable. \( A_k \) (an even an approximation to it) can change completely due to small perturbations in \( A \). Hard to make recursive sampling approaches work.
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Key Idea: Truncation ⇒ Regularization

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where \( \lambda = \frac{\|A - A_k\|_F^2}{k} \).
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- Ridge ‘washes out’ rather than completely removes contributions from small singular directions.
- These are just the standard leverage scores of $[A, \sqrt{\lambda}I]$!

Computable using the recursive sampling algorithms of [CLMMPS ’15].
Standard arguments show that sampling $\tilde{O}(k/\epsilon^2)$ columns by their ridge leverage scores gives an approximation:

$$(1 - \epsilon)AA^T - \epsilon \lambda I \preceq \tilde{A}\tilde{A}^T \preceq (1 + \epsilon)AA^T + \epsilon \lambda I.$$
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- We show that this is enough for $\tilde{\mathbf{A}}$'s top singular vector space to approximate that of $\mathbf{A}$.  

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- We show that this is enough for $\tilde{A}$'s top singular vector space to approximate that of $A$.
- Specifically, show $\tilde{A}$ is a good projection-cost-preserving sketch of $A$ [Cohen Elder Musco Musco Persu ’15].
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- We show that this is enough for $\tilde{A}$’s top singular vector space to approximate that of $A$.
- Specifically, show $\tilde{A}$ is a good projection-cost-preserving sketch of $A$ [Cohen Elder Musco Musco Persu ’15].
- Also achieve near optimal column subset selection via a connection between ridge scores and adaptive sampling [Deshpande Rademacher Vempala Wang ’06].
Low-Rank Approximation via Ridge Leverage Scores: Sampling $A$ using the leverage scores of $(A + \lambda I)$ give near optimal sized sketches for low-rank approximation.
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- Scores can be computed in input sparsity time via iterative approximation algorithms.
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• Scores can be computed in input sparsity time via iterative approximation algorithms.

Corollary: $O(\text{nnz}(A)) + \text{poly}(k, \epsilon)$ time to compute $\tilde{B}$ with:

$$\|A - \tilde{B}\|_F^2 \leq (1 + \epsilon) \min_{\|B\| = k} \|A - B\|_F^2$$
Why do we care about avoiding sparse random projections in the first place?
**Original Motivation:** Match $O(\text{nnz}(A))$ time random projection algorithms for matrix preconditioning and over-constrained linear regression.

- Li Miller Peng ’13
- Cohen Lee Musco Musco Peng Sidford ’15.
Reason #1: Sampling Preserves Structure and Sparsity.
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Even when $A$ is sparse, $\tilde{A} = AP$ will be dense. Limits compression for very sparse matrices.
Reason #1: Sampling Preserves Structure and Sparsity

Results for regression used in new work on sparsifying and solving Laplacian and SDD systems:

- Lee, Peng, Spielman ’15.
- Kyng, Lee, Peng, Sachdeva, Spielman ’16
- Jindal, Kolev ’16
Reason #2: Sampling works in settings where random projection does not apply.
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In this paper: Applications to single-pass streaming algorithms for the column subset selection problem.
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In follow up work:

· [Musco Musco ’16]: Linear time kernel matrix approximation.
· [Musco Woodruff ’16]: Sublinear time relative-error low-rank approximation of PSD matrices.
Sampling for kernels

\[ K_{i,j} = k(a_i, a_j), \quad K = \phi(A)^T \phi(A) \]
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- Working with full \( n \times n \) kernel matrix often prohibitive. Low-rank approximation is important for efficient kernel ridge regression, kernel PCA, kernel \( k \)-means clustering, etc.
Sampling for kernels

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- Working with full \( n \times n \) kernel matrix often prohibitive. Low-rank approximation is important for efficient kernel ridge regression, kernel PCA, kernel \( k \)-means clustering, etc.
- Sketching \( K \) directly requires \( \Omega(n^2) \) kernel evaluations.
How can we avoid this using sampling?
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\[ O(\text{dot products per level}) \sim O(\text{kernel evaluations}) \]

- Let's find a low-rank approximation for \( K_{1/2} = 2 \) without constructing all of \( K \).
How can we avoid this using sampling?

\[ O(nk) \text{ dot products per level} \sim O(nk) \text{ kernel evaluations} \]

1. Let us find a low-rank approximation for \( K_{1/2} \) without constructing all of \( K \).
2. \( A = K_{1/2} \)
How can we avoid this using sampling?

\[
O(\|
\begin{pmatrix} \mathbf{w} \end{pmatrix} \|
\|
\begin{pmatrix} \mathbf{x} \end{pmatrix} \|
\) \cdot\ \|
\begin{pmatrix} \mathbf{w} \\mathbf{x} \end{pmatrix} \|
\)
\approx\ O(\|
\begin{pmatrix} \mathbf{w} \end{pmatrix} \|
\|
\begin{pmatrix} \mathbf{x} \end{pmatrix} \|
\) \cdot\ \|
\begin{pmatrix} \mathbf{w} \\mathbf{x} \end{pmatrix} \|
\)

If we set \( \mathbf{A} = \mathbf{K}_1 = 2 \) so \( \mathbf{A}^\top \mathbf{A} = \mathbf{K} \).

Let's us find a low-rank approximation for \( \mathbf{K}_1 = 2 \) without constructing all of \( \mathbf{K} \).
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\[ O(nk) \text{ dot products per level} \sim O(nk) \text{ kernel evaluations if } A = K_{1,2} \text{ so } A A^T = K. \]

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Lets us find a low-rank approximation for \( K^1 = 2 \) without constructing all of \( K \).
How can we avoid this using sampling?

\[ O(nk^2) \text{ dot products per level} \approx O(nk^2) \text{ kernel evaluations if we set } A = K_{1/2} = \sqrt{2} \text{ so } AA^T = K \]

Let us find a low-rank approximation for \( K_{1/2} = \sqrt{2} \) without constructing all of \( K \).
How can we avoid this using sampling?

\[
\mathcal{O}(nk^2) \text{ dot products per level} \\
\sim \mathcal{O}(nk) \text{ kernel evaluations if we set } A = K_1 = 2 \\
\]

Let us find a low-rank approximation for \( K_1 = 2 \) without constructing all of \( K \).
How can we avoid this using sampling?

\[ O(nk) \text{ dot products per level } \Rightarrow \tilde{O}(nk) \text{ kernel evaluations if we set } A = K^{1/2} \text{ so } AA^T = K. \]
How can we avoid this using sampling?

- $O(nk)$ dot products per level $\Rightarrow \tilde{O}(nk)$ kernel evaluations if we set $A = K^{1/2}$ so $AA^T = K$.
- Lets us find a low-rank approximation for $K^{1/2}$ without constructing all of $K$. 

RECURSIVE SAMPLING
Summary: Input sparsity time linear algebra is not just about sparse random embeddings. Results can also achieved via leverage score sampling.
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Open Questions:

- Empirical evaluation, especially for kernel applications.
- Other methods of achieving input sparsity time? Deterministic?
- Further applications?