INPUT SPARSITY TIME LOW-RANK APPROXIMATION VIA RIDGE LEVERAGE SCORE SAMPLING

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- Supported by new results in random matrix theory and understanding of how to use these results algorithmically.
- Closely tied to work on graph sparsification, fast laplacian solvers, streaming algorithms, compressed sensing, etc.

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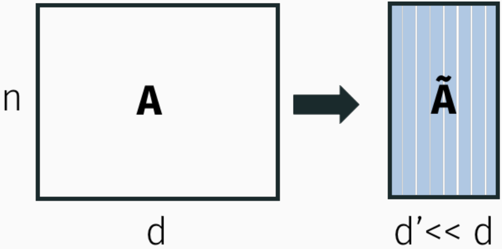

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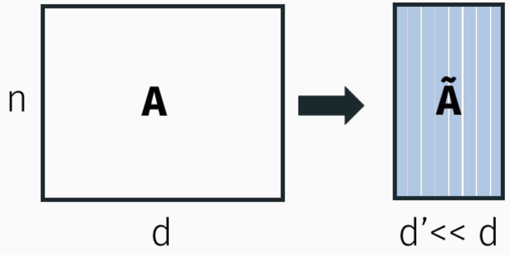

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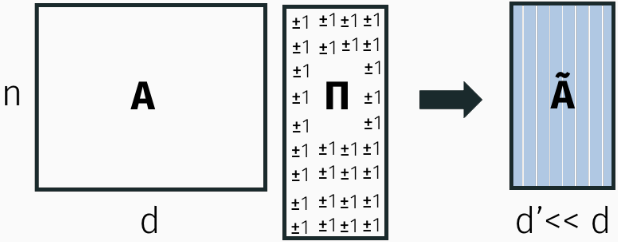

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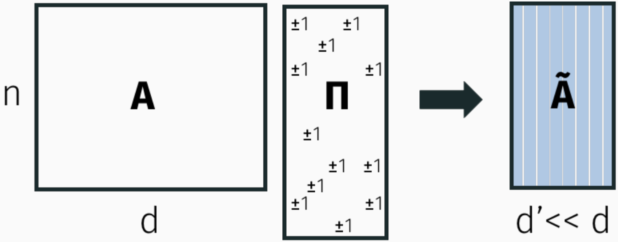

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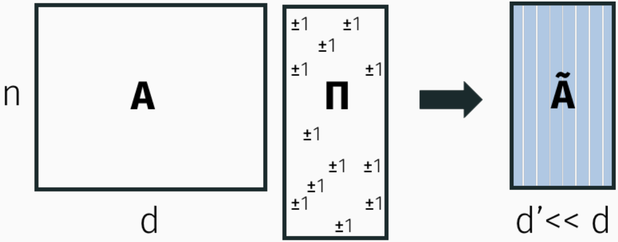
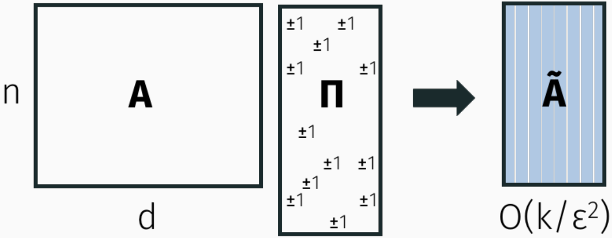
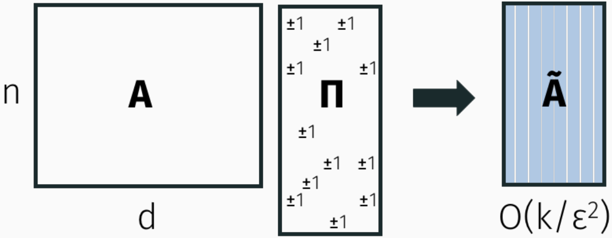


- Solution for $\tilde{\mathbf{A}} \Rightarrow$ approximate solution for $\mathbf{A}$ for problems like linear system solving, low-rank approximation, etc.
- $O(\text{nnz}(\mathbf{A}))$ to compute $\mathbf{A\Pi}$ plus lower order terms = input sparsity time



Set $\mathbf{Q} \leftarrow$ top k left singular vectors of $\tilde{\mathbf{A}}$.



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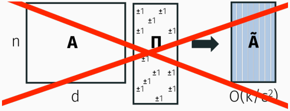
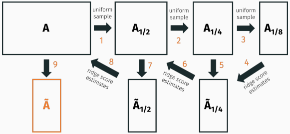
- Compare with $\tilde{O}(\text{nnz}(\mathbf{A}) \cdot k/\sqrt{\epsilon})$ for iterative methods.

Main Result: Input sparsity time low-rank approximation without sparse random projections.





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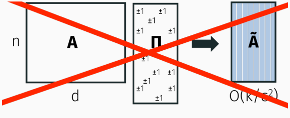

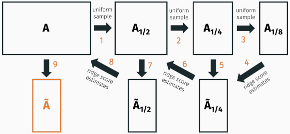

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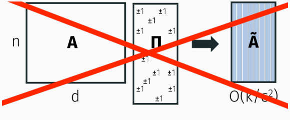

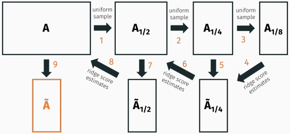

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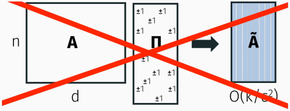

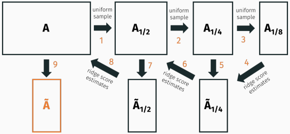

- Column subset selection in single-pass streams.
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- Sublinear time, relative error algorithms for low-rank approximation of PSD matrices [Musco Woodruff '16]

DIMENSIONALITY REDUCTION VIA IMPORTANCE SAMPLING

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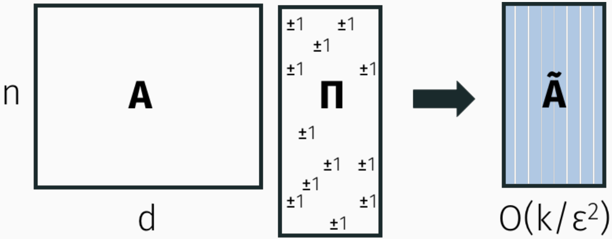

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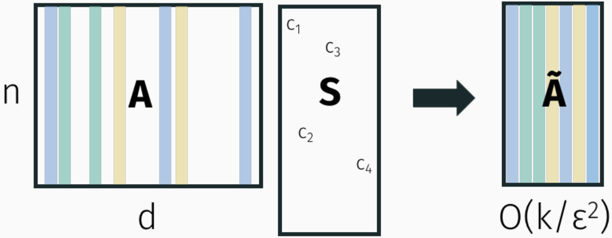
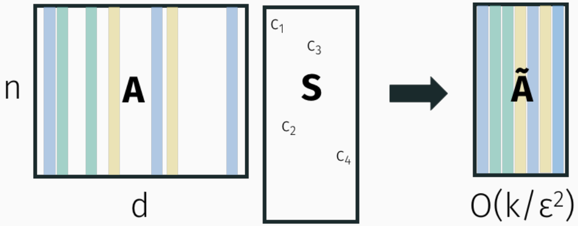


Extremely simple and efficient... once **S** is known.

Variations on statistical leverage scores give a sketch $\tilde{\mathbf{A}}$ that is sufficient for near-optimal low-rank approximation.



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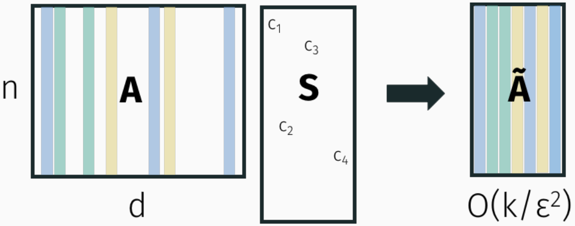

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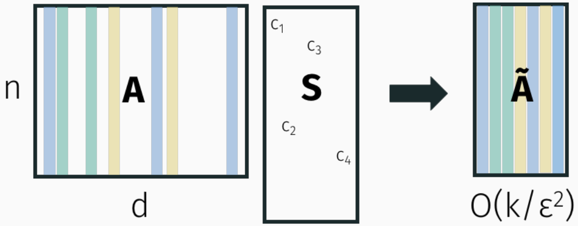

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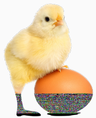

1. Brief discussion of techniques
2. Why care about sampling?

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Sampling $\tilde{O}(\text{rank}(\mathbf{A})/\epsilon^2)$ columns by leverage scores gives spectral approximation:

$$(1-\epsilon)\mathbf{A}\mathbf{A}^T \preceq \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T \preceq (1+\epsilon)\mathbf{A}\mathbf{A}^T.$$

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- Input sparsity time regression without sparse projections.

"Subspace Scores" [Drineas, Mahoney, Muthukrishnan '08], [Sarló '06]:

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- Gives additional error depending on $\|\mathbf{A} - \mathbf{A}_k\|_F^2 \implies$ good enough for near optimal low-rank approximation.

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- But this is what we want to compute in the first place! Hence all nnz($\mathbf{A}$) time sampling algorithms rely critically on sparse random projections.
- Further, subspace scores are unstable. $\mathbf{A}_k$ (an even an approximation to it) can change completely due to small perturbations in $\mathbf{A}$. Hard to make recursive sampling approaches work.

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- Ridge 'washes out' rather than completely removes contributions from small singular directions.
- These are just the standard leverage scores of $[\mathbf{A}, \sqrt{\lambda}\mathbf{I}]$! Computable using the recursive sampling algorithms of [CLMMPS '15].

Standard arguements show that sampling $\tilde{O}(k/\epsilon^2)$ columns by their ridge leverage scores gives an approximation:

$$(1-\epsilon)\mathbf{A}\mathbf{A}^T - \epsilon\lambda\mathbf{I} \preceq \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T \preceq (1+\epsilon)\mathbf{A}\mathbf{A}^T + \epsilon\lambda\mathbf{I}.$$

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- Specifically, show $\tilde{\mathbf{A}}$ is a good projection-cost-preserving sketch of $\mathbf{A}$ [Cohen Elder Musco Musco Persu '15].
- Also achieve near optimal column subset selection via a connection between ridge scores and adaptive sampling [Deshpande Rademacher Vempala Wang '06].

Low-Rank Approximation via Ridge Leverage Scores: Sampling $\mathbf{A}$ using the leverage scores of $(\mathbf{A} + \lambda \mathbf{I})$ give near optimal sized sketches for low-rank approximation.

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Corollary: $O(\text{nnz}(\mathbf{A})) + \text{poly}(k, \epsilon)$ time to compute $\widetilde{\mathbf{B}}$ with:

$$\|\mathbf{A}-\widetilde{\mathbf{B}}\|_F^2 \leq (1+\epsilon) \min_{\mathbf{B} \mid \text{rank}(\mathbf{B})=k} \|\mathbf{A}-\mathbf{B}\|_F^2$$

WHY SAMPLING?

Why do we care about avoiding sparse random projections in the first place?

Original Motivation: Match $O(\text{nnz}(\mathbf{A}))$ time random projection algorithms for matrix preconditioning and over-constrained linear regression.

- Li Miller Peng '13
- Cohen Lee Musco Musco Peng Sidford '15.

Reason #1: Sampling Preserves Structure and Sparsity.

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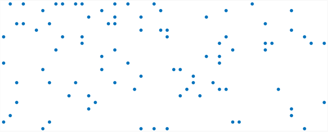
Original Data General Sketch Column Sample



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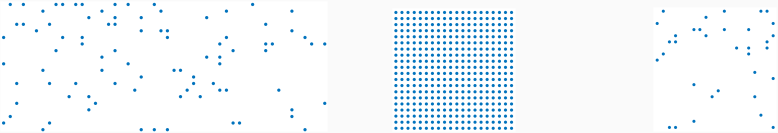

Even when $\mathbf{A}$ is sparse, $\tilde{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}$ will be dense. Limits compression for very sparse matrices.

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Results for regression used in new work on sparsifying and solving Laplacian and SDD systems:

- Lee, Peng, Spielman '15.
- Kyng, Lee, Peng, Sachdeva, Spielman '16
- Jindal, Kolev '16

Reason #2: Sampling works in settings where random projection does not apply.

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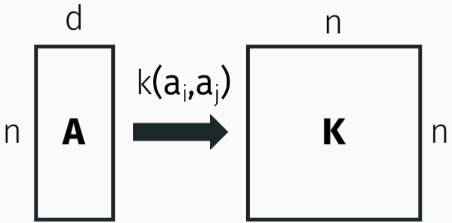
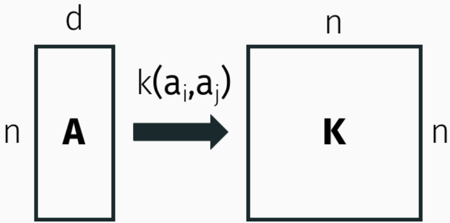
In follow up work:

- [Musco Musco '16]: Linear time kernel matrix approximation.
- [Musco Woodruff '16]: Sublinear time relative-error low-rank approximation of PSD matrices.



$$K_{i,j} = k(a_i, a_j),\ K = \phi(A)^T\phi(A)$$



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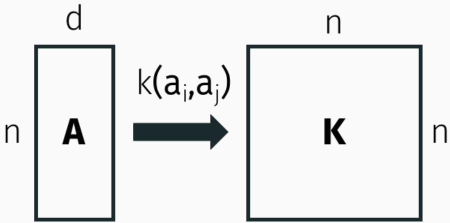

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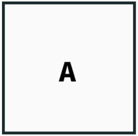
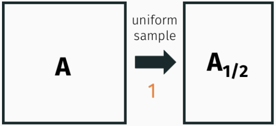


How can we avoid this using sampling?



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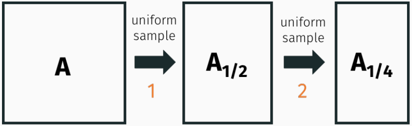

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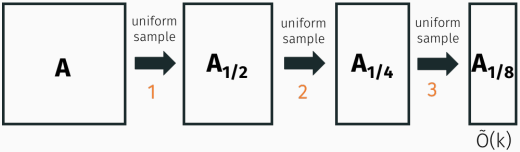

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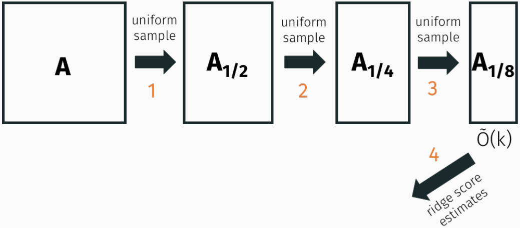

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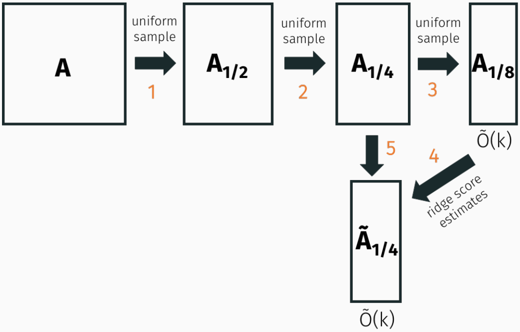

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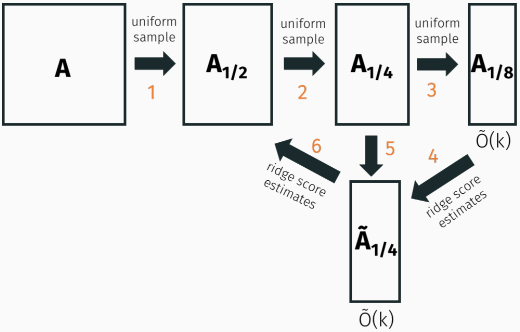

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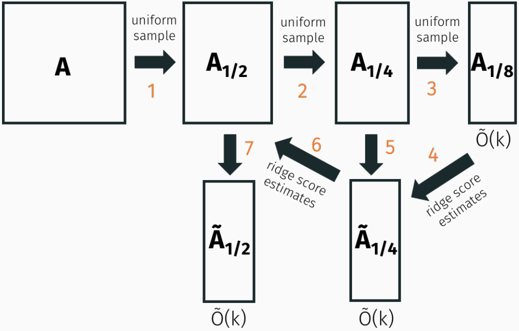

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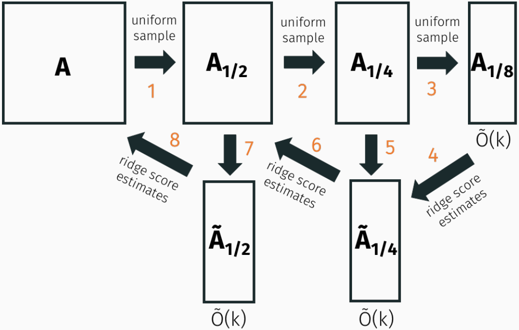

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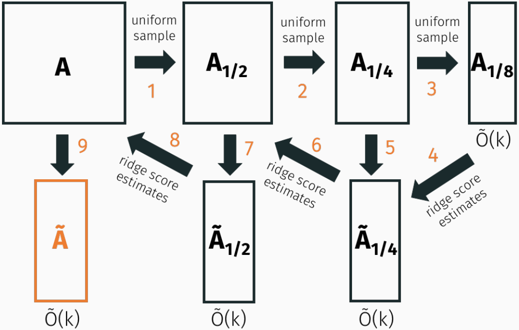

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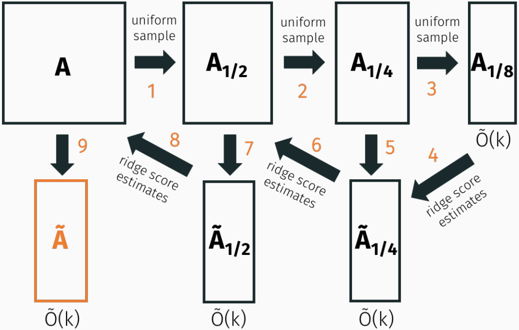

- $O(nk)$ dot products per level $\Rightarrow \tilde{O}(nk)$ kernel evaluations if we set $\mathbf{A} = \mathbf{K}^{1/2}$ so $\mathbf{A}\mathbf{A}^T = \mathbf{K}$.

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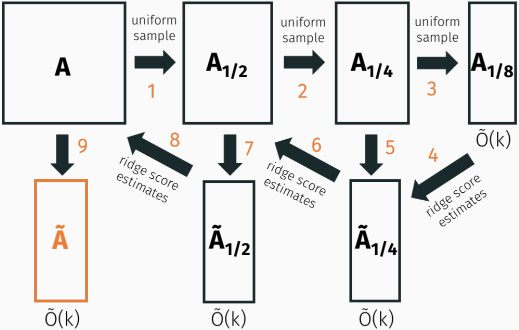

- $O(nk)$ dot products per level $\Rightarrow \tilde{O}(nk)$ kernel evaluations if we set $\mathbf{A} = \mathbf{K}^{1/2}$ so $\mathbf{A}\mathbf{A}^T = \mathbf{K}$.
- Lets us find a low-rank approximation for $\mathbf{K}^{1/2}$ without constructing all of $\mathbf{K}$.

Summary: Input sparsity time linear algebra is not just about sparse random embeddings. Results can also achieved via leverage score sampling.

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Open Questions:

- Empirical evaluation, especially for kernel applications.
- Other methods of achieving input sparsity time? Deterministic?
- Further applications?